Reflection group of the quaternionic Lorentzian Leech lattice

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Abstract: We study a second example of the phenomenon studied in the article "The complex Lorentzian Leech lattice and the bimonster". We find 14 roots in the automorphism group of the the quaternionic Lorentzian Leech lattice L that form the Coxeter diagram given by the incidence graph of projective plane over $\mathbb{F}_2$. We prove that the reflections in these 14 roots generate the automorphism group of L. The investigation is guided by an analogy with the the theory of Weyl groups. There is a unique point in the quaternionic hyperbolic space fixed by the "diagram automorphisms" that we call the Weyl vector. The unit multiples of the 14 roots forming the diagram are the analogs of the simple roots. The 14 mirrors perpendicular to the simple roots are the mirrors that are closest to the Weyl vector.

keywords: quaternionic Leech lattice, quaternionic hyperbolic reflection group, Y-groups, Weyl group, Coxeter diagram.

1. Introduction

Let $\mathcal{H}$ denote the ring of Hurwitz integers, consisting of the quaternions $(a+bi+cj+dk)/2$ where a, b, c and d are integers, all congruent modulo 2. Let Λ and E_8 be the Leech lattice and E_8 root lattice respectively considered as Hermitian lattices over $\mathcal{H}$. Let H be the 2 dimensional lattice $\mathcal{H} \oplus \mathcal{H}$ with gram matrix $\left(\begin{smallmatrix} 0 & 1+i \\ 1-i & 0 \end{smallmatrix} \right)$ and let $L = \Lambda \oplus H$. Reflection groups of this and other quaternionic Lorentzian lattices were studied by Allcock in [1] and [2].

In this article we investigate a second example of the phenomenon studied in the article [3] on complex Lorentzian Leech lattice and bimonster. Here the base ring $\mathbb{Z}[e^{2\pi i/3}]$ is replaced by the non-commutative ring $\mathcal{H}$, the incidence graph of $P^2(\mathbb{F}_3)$ is replaced by $P^2(\mathbb{F}_2)$ and there is a parallel story. For more on the bimonster and its presentations on the M_{666} and $\text{Inc}(P^2(\mathbb{F}_3))$ diagrams, see [5], [6] and [8].

We find 14 quaternionic reflections of order 4 in the reflection group of the Lorentzian quaternionic Leech lattice L that form the Coxeter diagram given by the incidence graph of the projective plane over $\mathbb{F}_2$. (When I told D. Allcock about it, I got to know that he had found this diagram too). This 14 node diagram D (see Fig. 1) is obtained by extending the M_{444} diagram which comes up naturally, as the lattice L is isomorphic to $3E_8 \oplus H$. (We shall describe an explicit isomorphism from $\Lambda \oplus H$ to $3E_8 \oplus H$ over $\mathcal{H}$ later in 2.5, because it is needed for our computations.) The three hands of the M_{444} diagram correspond to the three copies of E_8.

The main results are the following. We see that there is a "Weyl vector" for D in the quaternionic hyperbolic space that is fixed by the diagram automorphisms and that the reflections in the 14 roots of D generate the reflection group of L. Allcock showed that the reflection group has finite index in the automorphism group of L. We also see that the 14 mirrors of the reflections in the roots of D are the mirrors closest to the Weyl vector, and in that sense, the roots of D are the analog of the simple roots.

The definitions and proof here are often similar to the ones in [3], so we shall not mention them at each step. In view of the two examples, one would surely like to know whether

there are other examples of similar phenomenon for other Lorentzian lattices; and if a more conceptual meaning can be given to these diagrams in a suitably general context that would explain all the numerical coincidences seen in this and the previous article.

2. Preliminaries

2.1 Notation

All the notations are borrowed from the article [3] on complex Lorentzian leech lattice and are mostly consistent with the ones used in [1]. As the ring of Eisenstein integers $\mathcal{E}$ is replaced by the ring $\mathcal{H}$ of Hurwitz integers, the 14 node diagram D replaces the 26 node diagram, p replaces $\theta = \sqrt{-3}$, $\mathbb{F}_2$ replaces $\mathbb{F}_3$, et cetera.

D	The incidence graph of $p^2(\mathbb{F}_2)$
H	the hyperbolic cell over $\mathcal{H}$ with Gram matrix $\begin{pmatrix} 0 & \bar{p} \\ p & 0 \end{pmatrix}$
$\mathcal{H}$	The ring of Hurwitz integers, a copy of the D_4 root lattice sitting inside the quaternions.
$\mathbb{H}$	The skew field of real quaternions.
Λ	The Leech lattice as a 6 dimensional negative definite $\mathcal{H}$-lattice
p	$1 - i$
ϕ_r^α	the α-reflection in the vector r
ξ	$(1 + i)/\sqrt{2}$

2.2 The ring of quaternions

let $\mathcal{H}$ be the ring of *Hurwitz integers* generated over $\mathbb{Z}$ by the 24 unit quaternions $\pm 1, \pm i, \pm j, \pm k$, and $(\pm 1 \pm i \pm j \pm k)/2$. The ring $\mathcal{H}$ consists of the elements $(a + bi + cj + dk)/2$, where a, b, c, d are integers all congruent modulo 2, with the standard multiplication rules $i^2 = j^2 = k^2 = ijk = -1$. When tensored with $\mathbb{R}$ we get the skew field of quaternions called $\mathbb{H}$. The conjugate of $q = a + bi + cj + dk$ is $\bar{q} = a - bi - cj - dk$. The *real part* of q is $\mathrm{Re}(q) = a$ and the *imaginary part* is $\mathrm{Im}(q) = q - \mathrm{Re}(q)$. The *norm* of q is $|q|^2 = \bar{q}.q$.

We let $\alpha = (1 + i + j + k)/2$ and $p = (1 - i)$. In the constructions of the lattices given below, the number $p = 1 - i$ plays the role of $\sqrt{-3}$ in [3]. p generates a two sided ideal $\mathfrak{p}$ in $\mathcal{H}$. We have $i \equiv j \equiv k \equiv 1 \bmod \mathfrak{p}$. (Observe that $(1 - i)\alpha = 1 + j$, so $j \equiv 1 \bmod \mathfrak{p}$). The group $\mathcal{H}/\mathfrak{p}\mathcal{H}$ is $\mathbb{F}_4$, generated by $0, 1, \alpha, \bar{\alpha}$.

The multiplicative group of units $\mathcal{H}^*$ is $2 \cdot A_4$. The quotient $\mathcal{H}^*/\{\pm 1\}$ has four Sylow three subgroups generated by $\alpha, i\alpha, j\alpha, k\alpha$. The permutation representation of $\mathcal{H}^*/\{\pm 1\}$ on the Sylow 3-subgroups identifies it with the alternating group A_4.

We also identify the quaternions $a + bi$ with the complex numbers. So any quaternion can be written as $z_1 + z_2 j$ for complex numbers z_1 and z_2. The multiplication is defined by $j^2 = -1$ and $jz = \bar{z}j$. The complex conjugation becomes conjugation by the element j.

2.3 Lattices over Hurwitz integers

A general reference for lattices is [7]. An $\mathcal{H}$-*lattice* is a free finitely generated right $\mathcal{H}$-module with an $\mathcal{H}$-valued bilinear form $\langle\ ,\ \rangle$ satisfying $\overline{\langle x, y \rangle} = \langle y, x \rangle$, $\langle x, y\alpha \rangle = \langle x, y \rangle \alpha$ and $\langle x\alpha, y \rangle = \bar{\alpha}\langle x, y \rangle$, for all x, y in the lattice and α in $\mathcal{H}$. In this article, by a lattice we shall mean an $\mathcal{H}$-lattice, unless otherwise stated. Definite lattices will usually be negative definite. The standard negative definite lattice $\mathcal{H}^n$ has the inner product $\langle x, y \rangle = -\bar{x}_1 y_1 - \cdots - \bar{x}_n y_n$. The indefinite lattice given by $\mathcal{H} \oplus \mathcal{H}$ with inner product $\langle (x, y), (x', y') \rangle = (\bar{x}, \bar{y})\left(\begin{smallmatrix} 0 & \bar{p} \\ p & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right)$ is denoted by H. We call H the *hyperbolic cell*.

Let x be a vector in a $\mathcal{H}$-lattice K. The norm $|x|^2 = \langle x, x \rangle$ is a rational integer. The $\mathbb{Z}$-module K with the quadratic form given by the norm will be called the *real form* of K. For example, the real form of the lattice $\mathcal{H}$ (with the norm multiplied by a factor of 2) is the D_4 root lattice.

The E_8 root lattice can be defined as a sub-lattice of $\mathcal{H}^2$ as $E_8 = \{(x_1, x_2) | x_1 \equiv x_2 \bmod \mathfrak{p}\}$. It has minimal norm -2 and the underlying $\mathbb{Z}$-lattice is the usual E_8 root lattice.

The Leech lattice Λ can be defined as a 6 dimensional negative definite $\mathcal{H}$-lattice with minimal norm -4 whose real form is the usual real Leech lattice. The automorphism group of this lattice was studied by Wilson in [11]. We quote the facts we need from there. Let $\omega = (-1 + i + j + k)/2$. The lattice Λ consists of all vectors $(v_\infty, v_0, v_1, v_2, v_3, v_4)$ in $\mathcal{H}^6$ such that $v_2 \equiv v_3 \equiv v_4 \bmod \mathfrak{p}$, $(v_1 + v_4)\bar{\omega} + (v_2 + v_3)\omega \equiv (v_0 + v_1)\omega + (v_2 + v_4)\bar{\omega} \equiv 0 \bmod 2$, and $-v_\infty(i + j + k) + v_0 + v_1 + v_2 + v_3 + v_4 \equiv 0 \bmod 2 + 2i$. The inner product we use is $-1/2$ of the one used in [11], so that the following basis vectors have norm -4. We use the following $\mathcal{H}$-basis for the lattice Λ given in [11] for some computations.

$$bb[1] = [2 + 2i, 0, 0, 0, 0, 0]$$
$$bb[2] = [2, 2, 0, 0, 0, 0]$$
$$bb[3] = [0, 2, 2, 0, 0, 0]$$
$$bb[4] = [i + j + k, 1, 1, 1, 1, 1]$$
$$bb[5] = [0, 0, 1 + k, 1 + j, 1 + j, 1 + k]$$
$$bb[6] = [0, 1 + j, 1 + j, 1 + k, 0, 1 + k]$$

Let L be the Lorentzian lattice $L = \Lambda \oplus H \cong 3E_8 \oplus H$. The real form of this lattice is $II_{4,28}$. E_8, Λ, H and L each satisfy $L'p = L$, where L' is the *dual lattice* of L defined by $L' = \{x \in L \otimes \mathbb{H} : \langle x, y \rangle \in \mathcal{H} \forall y \in L\}$.

2.4 Quaternionic reflections

A μ-*reflection* in a vector r of a lattice is given by

$$\phi_r^\mu(v) = v - r(1 - \mu)\langle r, v \rangle / |r|^2 \tag{1}$$

where $\mu \neq 1$ is a unit in $\mathcal{H}$. Note that

$$\phi_{r\alpha}^\mu = \phi_r^{\alpha\mu\alpha^{-1}} \tag{2}$$

for any unit $\alpha \in \mathcal{H}$. Note also that ϕ_r^μ can be characterized as the automorphism of the lattice that fixes the orthogonal complement of r and multiplies r by the root of unity μ. It follows that for every automorphism γ of the lattice we have $\gamma\phi_r^\mu\gamma^{-1} = \phi_{\gamma r}^\mu$. We say the two reflections ϕ_1 and ϕ_2 *braid* if $\phi_1\phi_2\phi_1 = \phi_2\phi_1\phi_2$. A *root* of a negative definite or a Lorentzian

lattice is a lattice vector of negative norm such that there is a nontrivial reflection in it that is an automorphism of the lattice. If $L'p = L$, the roots of L are all the vectors of norm -2. For such a root r, $R(L)$ contains six reflections of order four given by $\phi_r^{\pm i}, \phi_r^{\pm j}$ and $\phi_r^{\pm k}$. The square of each of them is the order 2 reflection ϕ_r^-. Note that the six units $\pm i, \pm j$, and $\pm k$ form a conjugacy class in $\mathcal{H}^*$. For a set of reflections (or roots) that either braid or commute, we form the *Coxeter diagram* by taking one vertex for each reflection and joining them only if the reflections braid.

For computational purpose we note the following (one needs to be careful because $\mathcal{H}$ is not commutative). Let $e_1, \cdots, e_n$ be a basis for a lattice. For a linear transformation ϕ, if $\phi(e_t) = \sum_s e_s \phi_{st}$ then let $mat(\phi) = ((\phi_{st}))$; if a vector $x = \sum e_t x_t$ is written in coordinates as a column vector then $\phi(x) = mat(\phi)(x_1, \cdots, x_n)'$.

2.5 Computation for explicit isomorphism between $3E_8 \oplus H$ and $\Lambda \oplus H$ over $\mathcal{H}$

The reflection group of the $\mathcal{H}$-lattice E_8 can be generated by two reflections of order 4 that braid with each other. So an E_8-diagram for us, looks like an A_2 Dynkin diagram. We say that two diagrams are orthogonal if the roots in the first diagram are orthogonal to the roots in the second diagram. We need to find three orthogonal E_8-diagrams in the lattice $\Lambda \oplus H$ and a hyperbolic cell orthogonal to this $3E_8$, to get an explicit change of basis matrix. We find the 6 vectors of the form $r = (l; 1, \bar{p}^{-1}(1 + \beta))$ (with $\beta \in \operatorname{Im} \mathbb{H}$ and l in the first shell of the leech lattice) in $\Lambda \oplus H$ forming three copies of E_8 by a computer search. The steps of the computation are given below.

First, note that i-reflections in two roots $r = (l; 1, \bar{p}^{-1}(1 + \beta))$ and $r' = (l'; 1, \bar{p}^{-1}(1 + \beta'))$ will commute if $|l - l'|^2 = -4$ and $\beta - \beta' = [l, l']$, where $[l, l'] = \operatorname{Im}\langle l, l' \rangle$. The i-reflections in r and r' will braid if $|l - l'|^2 = 6$ and $\beta - \beta' = [l, l'] \pm i$.

It is easy to find one copy of E_8. One just have to find two vectors l_1, l_2 in first shell of Leech lattice that are at a distance -6 and find β_1, β_2 accordingly.

Now we generate a large list of vectors in the first shell of Leech lattice using the basis bb given in 2.3. We find all the vectors l in the first shell (actually from the almost complete list that we had), that might give a root $(l; 1, *)$ of an orthogonal E_8. The conditions that l has to satisfy are $|l - l_1|^2 = |l - l_2|^2 = -4$ and $[l_1, l_2] - [l_1, l] + [l_2, l] = i$. This gives a small list of vectors l. (It is amusing to note that the equation looks like a "co-cycle condition").

Next, find pairs of vectors l_3, l_4 from the previous list that can actually give an E_8 orthogonal to the first one. The conditions to be satisfied are $[l_1, l_2] - [l_3, l_4] + [l_2, l_4] - [l_1, l_3] = 0$ and $|l_3 - l_4|^2 = -6$. This way one can find two E_8 diagrams orthogonal to the first one, that happen to be orthogonal to each other too. Thus we get the 6 vectors forming a $3E_8$ diagram.

Now we take the orthogonal complement and find two norm zero vectors in the complement forming an hyperbolic cell.

The rows of the matrix given below are the actual coordinates for the 8 vectors found by above calculation. The first six root vectors form a basis for the lattice $3E_8$ and the last two
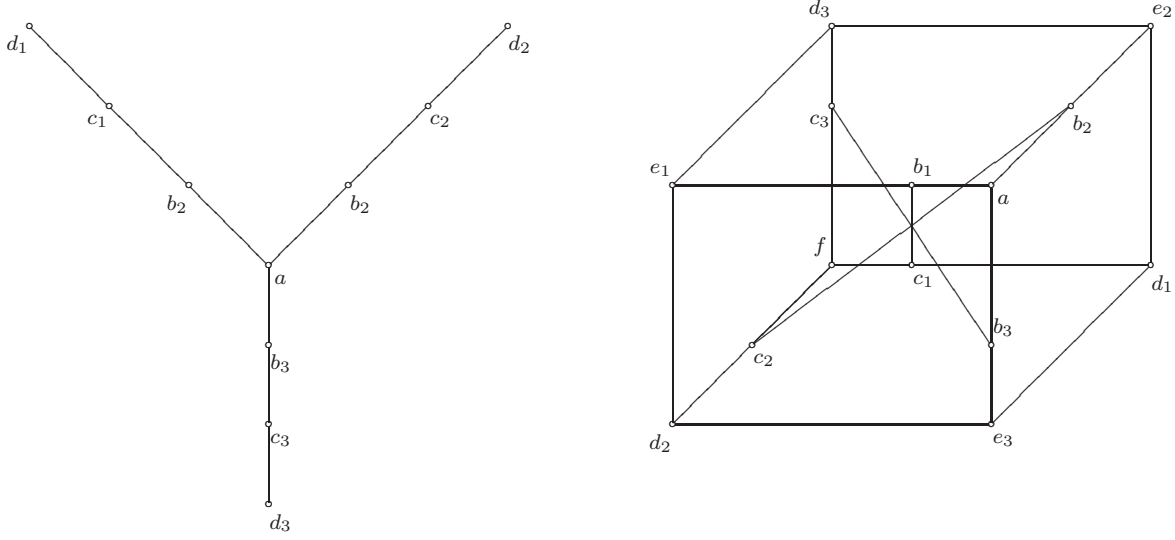
FIGURE 1. The diagrams M_{444} and D

norm zero vectors form a basis of a hyperbolic cell orthogonal to the first six vectors.

$$\begin{pmatrix}
2 & 2 & 0 & 0 & 0 & 0 & 1 & 1 \\
i+j+k & 1 & 1 & 1 & 1 & 1 & 1 & 2+i+j \\
\frac{1+i+j+k}{2} & \frac{3+i-j+k}{2} & \frac{1-i+j-k}{2} & \frac{1-i-j+k}{2} & \frac{-1-i-j-k}{2} & \frac{1-i-j+k}{2} & 1 & \frac{3+i+j+k}{2} \\
1+i+k & 1 & k & -k & 1 & -j & 1 & \frac{3+i+j+k}{2} \\
1+i+k & 1 & i & -j & -i & 1 & 1 & \frac{3+i+j+k}{2} \\
1+i & 1+k & 1-i & 1-k & 0 & 0 & 1 & \frac{3+i+j+k}{2} \\
\frac{-7+i-3j-5k}{2} & \frac{-7+3i-3j+k}{2} & \frac{-1+3i-j+k}{2} & \frac{-1+i-j+3k}{2} & \frac{-1+3i+j+3k}{2} & \frac{-3+i-j+k}{2} & \frac{-5+3i-3j+k}{2} & -4+2i-3j-k \\
\frac{-5-3i-j-5k}{2} & \frac{-7+i-j-3k}{2}, & \frac{-1+i-j-k}{2} & \frac{-3+i-j+k}{2} & \frac{-1+3i-j+k}{2} & \frac{-3+i+j-k}{2} & \frac{-5+i-j-k}{2} & -4-2j-2k
\end{pmatrix}$$

3. THE 14 NODE DIAGRAM

3.1 the diagram of 14 roots

The i-reflection of order 4 in a root r braids with the i-reflection in a root r' if $\langle r, r' \rangle = p$. In this section we work in the co-ordinates $3E_8 \oplus H$ for L. We can find 10 roots a, b_s, c_s, d_s for $s = 1, 2, 3$ forming an M_{444} diagram inside the reflection group $R(L)$. See [8] for more on these groups (called Y-groups there). The hands of the M_{444}-diagram correspond to the three copies of E_8 in L, in the sense that c_s, d_s generate the reflection group of the s-th E_8, the b_s are the affinizing node and a is the hyperbolizing node. These 10 roots can be extended to a set of 14 roots forming the incidence graph of $\mathbb{P}^2(\mathbb{F}_2)$. The roots a, c_s, e_s correspond to the points of $\mathbb{P}^2(\mathbb{F}_2)$ (See Fig. 1) and f, b_s, d_s correspond to the lines. There is an edge between them if the point lies on the line. This 14 vertex diagram is called D. the seven roots a, c_s, e_s (or their unit multiples) are called "points" and the seven roots f, b_s, d_s (or their unit multiples) are called "lines". One choice of the explicit coordinates of these roots

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are given by

$$
\begin{array}{rll}
r[1] = & a = [& , & ; \quad 1 \quad , \quad -1 \quad] \\
r[1+s] = & c_s = [\;\; (1,-1)_s \;\; , & & ; \quad \quad , \quad \quad] \\
r[4+s] = & e_s = [& (1,1)_{tu} \;\; ; & -i \quad , \quad -1 \quad] \\
r[8] = & f = [\;\; (\;\; ,p)^3 & ; & -ip \;\; , \quad -p \quad] \\
r[8+s] = & b_s = [\;\; (\;\; ,p)_s \;\; , & & ; \quad -1 \quad , \quad \quad] \\
r[11+s] = & d_s = [\;\; (-p, \;\;)_s \;\; , & & ; \quad \quad , \quad \quad]
\end{array}
$$

An E_8 vector with subscript s means we put it in place of the s-th E_8, while the subscript tu means that we put it at the t-th and u-th place. The indices s, t and u are in cyclic permutation of $(1, 2, 3)$. Blank spaces are to be filled with zero.

3.2 Linear relations among the roots of D

Let x be a point and l be a line of D. $\langle x, l \rangle$ is equal to p or 0 depending on whether there is a edge between x and l or not. Using this we find that the vector

$$
w_{\mathcal{P}} = l\bar{p} + \sum_{x \in l} x \tag{3}
$$

is perpendicular to the points and has norm 2. But there is only one such vector. Hence we get the relations $l\bar{p} + \sum_{x \in l} x = l'\bar{p} + \sum_{x \in l'} x$. for any two lines l and l'. These relations generate all linear relations in the 14 roots. Using the automorphism σ that takes l to x and x to li we see that the element

$$
w_{\mathcal{L}} = xp + \sum_{x \in l} l \tag{4}
$$

is constant for all point x. We shall see that the elements $w_{\mathcal{P}}$ and $w_{\mathcal{L}}$ determine points in the quaternionic hyperbolic space fixed by the diagram automorphisms $PGL_2(\mathbb{F}_2)$.

4. Generators for the reflection group of L

4.1 The Heisenberg group

We follow the definitions and notations of section 6 of the article [1]. Let $\mathbb{T}$ be the *Heisenberg group* generated by the translations $T_{\lambda,z}$ for every λ in $\Lambda = \Lambda \cap \Lambda' p$. In section 6 Allcock mentions that formulae (3.2) - (3.5) of [1] holds in the quaternionic case too. We need to do a little more calculation to get a little stronger version of Theorem 6.1 of [1]. We use the three roots $r_1 = (0^6; 1, i)$, $r_2 = (0^6; 1, -1)$ and $r_3 = (0^6; 1, -\epsilon)$ where $\epsilon = (1 - i - j + k)/2$.

4.2. Lemma. *Let $\epsilon = (1 - i - j + k)/2$, r_3 and r_2 be the roots $(0^6; 1, -\epsilon)$ and $(0^6; 1, -1)$ and let $R = \phi_{r_3}^i \phi_{r_2}^j$. Then*

$$
T_{\lambda,z}^{-1} R T_{\lambda,z} R^{-1} = T_{\lambda(\bar{\epsilon}-1), \epsilon z \bar{\epsilon} - z + \mathrm{Im}\langle \lambda \bar{\epsilon}, -\lambda \rangle} \tag{5}
$$

Since $(\bar{\epsilon} - 1)$ is an unit of $\mathcal{H}$ we see that for every $\lambda \in \Lambda$, $R(L)$ contains a reflection in λ.

Proof. R is of the form $\left(\begin{smallmatrix} I & 0 \\ 0 & R_H \end{smallmatrix} \right)$ where $R_H = \left(\begin{smallmatrix} \epsilon & 0 \\ u & \delta \end{smallmatrix} \right)$ is the matrix acting on H with $u = (3 - i + j - k)/2$ and $\delta \bar{p}^{-1} = \bar{p}^{-1} \epsilon$. Now $R_H^{-1} = \left(\begin{smallmatrix} \epsilon^{-1} & 0 \\ v & \delta^{-1} \end{smallmatrix} \right)$ where $v = -\delta^{-1} u \epsilon^{-1}$.

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Now just multiply the matrices to see that

$$
RT_{\lambda,z}R^{-1} = \begin{pmatrix} I & & \\ & \epsilon & \\ & u & \delta \end{pmatrix} \begin{pmatrix} I & & \lambda \\ & & 1 \\ -\bar{p}^{-1}\lambda^* & \bar{p}^{-1}(z-\lambda^2/2) & 1 \end{pmatrix} \begin{pmatrix} I & & \\ & \bar{\epsilon} & \\ & v & \bar{\delta} \end{pmatrix}
$$

$$
= \begin{pmatrix} I & & \lambda \\ & & \epsilon \\ -\delta\bar{p}^{-1}\lambda^* & u+\delta\bar{p}^{-1}(z-\lambda^2/2) & \delta \end{pmatrix} \begin{pmatrix} I & & \\ & \bar{\epsilon} & \\ & v & \bar{\delta} \end{pmatrix}
$$

$$
= \begin{pmatrix} I & & \lambda\epsilon^{-1} \\ & & 1 \\ -\delta\bar{p}^{-1}\lambda^* & x & 1 \end{pmatrix}
$$

where

$$
x = (u + \delta\bar{p}^{-1}(z-\lambda^2/2))\bar{\epsilon} + \delta v = \delta\bar{p}^{-1}(z-\lambda^2/2)\bar{\epsilon} = \bar{p}^{-1}\epsilon(z-\lambda^2/2)\bar{\epsilon} = \bar{p}^{-1}(\epsilon z\bar{\epsilon} - \lambda^2/2)
$$

In the second equality we use $u\bar{\epsilon} = -\delta v$ and in the third $\delta\bar{p}^{-1} = \bar{p}^{-1}\epsilon$. Also note that $-\delta\bar{p}^{-1}\lambda^* = -\bar{p}^{-1}\epsilon\lambda^* = -\bar{p}^{-1}(\lambda\bar{\epsilon})^*$. Thus we have

$$
RT_{\lambda,z}R^{-1} = T_{\lambda\bar{\epsilon},\epsilon z\bar{\epsilon}} \tag{6}
$$

Now (5) follows from (6) and $T_{\lambda,z}T_{\lambda',z'} = T_{\lambda+\lambda',z+z'+\mathrm{Im}\langle\lambda',\lambda\rangle}$ (equation (3.2) in [1]). $\qquad\square$

4.3. Lemma. *Let Ψ be the set of roots that are unit multiples of the roots of the form $(\lambda; 1, *)$. The reflections in Ψ act transitively on all roots. The reflections in all the roots of the form $(\lambda, 1, *)$ generate the reflection group of L.*

Proof. The calculation here is almost identical to Theorem 6.2 in [1] which uses the idea in [4]. For this lemma only, let $h(\lambda; \mu, \eta) = |\mu|$. (Later we are going to use a different definition of height). The roots r in Ψ are the ones with $h(r) = 1$. We show that if we have a root r with $h(r) > 1$, then we can i-reflect it in a root of Ψ to decrease its height. The covering radius of the Leech lattice is used in this proof and it has just the right value to make things work.

Let $y = (l; 1, \bar{p}^{-1}(\alpha - l^2/2))$ be a multiple of a root r with $h(r) > 1$. We have $|y|^2 \in (-2, 0)$, which amount to $\mathrm{Re}(\alpha) \in (-1, 0)$ because $|y|^2 = 2\mathrm{Re}(\alpha)$. Consider the i-reflection in the root $r = (\lambda; 1, \bar{p}^{-1}(-1-\lambda^2/2+\beta+n))$; where $\beta \in \mathrm{Im}\,\mathbb{H}$ is determined so that $\bar{p}^{-1}(-1-\lambda^2/2+\beta)$ is in $\mathcal{H}$ and n (to be chosen later) can be any element of $\mathrm{Im}\,\mathfrak{p} = \mathfrak{p} \cap \mathrm{Im}\,\mathbb{H}$. Calculation yields $\langle r, y \rangle = -2(a+b)$, where

$$
-2a = -\frac{1}{2}|l-\lambda|^2 - 1 + \mathrm{Re}(\alpha) \in \mathbb{R}
$$

and

$$
-2b = \mathrm{Im}(\alpha) + \mathrm{Im}\langle\lambda, l\rangle - \beta - n \in \mathrm{Im}\,\mathbb{H}
$$

So $h(\phi_r^i(y)) = \bar{p}(1 - (1-i)(a+b))$. Thus we want to make $|1 - (1-i)(a+b)|^2 < 1$, which amount to

$$
|1/2 - a|^2 + |i/2 - b|^2 < 1/2 \tag{7}
$$

Because the covering norm of Leech lattice is 2 we can make $|l - \lambda|^2 \in [-2, 0]$. This, together with $\mathrm{Re}(\alpha) \in (-1, 0)$ gives $a \in (0, 1)$. So $|1/2 - a|^2 < 1/4$. As for the second term of (7), $(i - 2b)$ is in $\mathrm{Im}\,\mathbb{H}$, and in the expression for $-2b$ we are free to choose $n \in \mathrm{Im}\,\mathfrak{p}$ which forms a copy of D_3 root lattice: $\{(ai + bj + ck) : a+b+c \equiv 0 \bmod 2\}$ in $\mathrm{Im}\,\mathbb{H}$. The covering radius

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of D_3 is 1. So we can make the norm of $(i - 2b)$ less than 1 by choice of n and thus make $|i/2 - b|^2 \leq 1/4$.

So if ϕ_r^μ is a reflection in any root r, after conjugating finitely many times by i-reflections in roots of the form $(\lambda; 1, *)$ we get a reflection $\phi_{r'}^\zeta$ in a root $r' = r''u$ where $r'' = (\lambda''; 1, *)$ and u is an unit. But then $\phi_{r'}^\zeta = \phi_{r''}^{u\zeta u^{-1}}$. Thus ϕ_r^μ can be obtained as a product of reflections in the roots of the form $(\lambda; 1, *)$. $\qquad\square$

4.4. Lemma. *Let $\lambda_1, \lambda_2, \cdots, \lambda_{24}$ be elements of Λ that make a $\mathbb{Z}$-basis. Let $r_1 = (0^6; 1, i)$ and r_2, r_3 be as given in Lemma 4.2. Fix z_s such that $T_{\lambda_s, z_s} \in \mathbb{T}$, for $s = 1, \cdots, 24$. Let R_1 temporarily denote the group generated by all the reflections in the 81 roots $T_{\lambda_s, z_s}(r_t)$, $T_{0, i+j}(r_t)$, $T_{0, i+k}(r_t)$ and r_t where $s = 1, \cdots, 24$ and $t = 1, 2, 3$. Then R_1 contains the Heisenberg group $\mathbb{T}$. In fact R_1 is equal to the reflection group of L.*

Proof. From (5) in lemma 4.2 we get that for each λ_s, the group R_1 contains a translation in the vector $\lambda_s(\bar{\epsilon} - 1)$. These vectors form a $\mathbb{Z}$-basis of L as $\bar{\epsilon} - 1$ is an unit. Using $T_{\lambda, z} \circ T_{\lambda', z'} = T_{\lambda + \lambda', z + z' + \mathrm{Im}\langle\lambda', \lambda\rangle}$ and $T_{\lambda, z}^{-1} = T_{-\lambda, -z}$ (equations (3.2) and (3.3) in [1]) we see that R_1 contains translation in every vector of Λ.

Now we argue that all the central translations of the form $T_{0, z}$ are in R_1. Choosing λ and λ' such that $\langle\lambda', \lambda\rangle = p$ and using the identity $T_{\lambda, z}^{-1} T_{\lambda', z'}^{-1} T_{\lambda, z} T_{\lambda', z'} = T_{0, 2\,\mathrm{Im}\langle\lambda', \lambda\rangle}$ (equation (3.4) in [1]) conclude that the central translation $T_{0, 2i}$ is in R_1. Similarly taking $\langle\lambda', \lambda\rangle$ to equal $p\alpha = (1 + j)$ and $p(1 + i - j + k)/2 = (1 + k)$ respectively, it follows that the central translation $T_{0, 2j}$ and $T_{0, 2k}$ are also in R_1. From (6) in 4.2, it follows that $T_{0, z}^{-1} R T_{0, z} R^{-1} = T_{0, \epsilon z \bar{\epsilon} - z}$ which equals $T_{0, -i-2j-k}$ for $z = i + j$ and $T_{0, -j-k}$ for $z = i + k$. So these central translations are in R_1 too. We found that $T_{0, i+k}$ and $T_{0, j+k}$ are in R_1, as are $T_{0, 2i}, T_{0, 2j}$ and $T_{0, 2k}$. These central translations clearly generate all the central translations of the form $T_{0, z}$, $z \in \mathfrak{p}\mathcal{H}$. So R_1 contains $\mathbb{T}$.

the orbit of $(0^6; 1, -1)$ under $\mathbb{T}$ is all roots of the form $(\lambda, 1, \bar{p}^{-1}(\beta - 1 - \lambda^2/2))$. So all these roots are in R_1 and we have already seen that these generate the whole reflection group of L.

For the actual computation multiply the basis vectors bb given in 2.3 by $1, i, j, (-1 + i + j + k)/2$ to get a basis of Λ over $\mathbb{Z}$. $\qquad\square$

5. The fixed points of diagram automorphisms and height of a root

5.1 The fixed points under diagram automorphism

The group $PGL_2(\mathbb{F}_2)$ acts on the diagram D and this induces a linear action of $PGL_2(\mathbb{F}_2)$ on L. The graph automorphism switching points with lines in $Inc(\mathbb{P}^2(\mathbb{F}_2))$ lifts to a automorphism σ taking a line l to a point x and x to li (note: $\langle x, l\rangle = p$ if and only if $\langle li, x\rangle = p$). This gives action of the extended diagram automorphism group $Q = 8 \cdot PGL_2(\mathbb{F}_2)$ on L and hence on the *quaternionic hyperbolic space* $\mathbb{H}H^7$, which consists of the positive norm lines in $L \otimes \mathbb{R}$. The vectors $w_{\mathcal{P}}$ and $w_{\mathcal{L}}$ defined in (3) and (4) of section 3.2 span the 2-dimensional space fixed by the action of $PGL_2(\mathbb{F}_2)$. From (3) and (4) it follows that $\sigma(w_{\mathfrak{p}}) = w_{\mathcal{L}}i$ and $\sigma(w_{\mathcal{L}}) = w_{\mathcal{P}}$.

Let $\Sigma_{\mathcal{P}}$ and $\Sigma_{\mathcal{L}}$ be the sum of the points and the sum of the lines respectively. These too are fixed by the $PGL_2(\mathbb{F}_2)$ action and σ takes $\Sigma_{\mathcal{L}}$ to $\Sigma_{\mathcal{P}}$ and $\Sigma_{\mathcal{P}}$ to $\Sigma_{\mathcal{L}}i$. So, there is a unique fixed point in $\mathbb{H}H^7$ under the action of this extended group of diagram automorphisms Q,

given by the image of the vector $\Sigma_{\mathcal{P}} + \Sigma_{\mathcal{L}}\xi$ or $w_{\mathcal{P}} + w_{\mathcal{L}}\xi \in L \otimes \mathbb{R}$ where $\xi = (1+i)/\sqrt{2}$ is an eighth root of unity: $\xi^2 = i$. We call this fixed vector the *Weyl vector* :

$$\bar{\rho} = (\Sigma_{\mathcal{P}} + \Sigma_{\mathcal{L}}\xi)/14 \tag{8}$$

We note some of the inner products between the special vectors that we need later : Let $(\rho_1, \cdots, \rho_{14}) = (x_1, \cdots, x_7, l_1\xi, \cdots, l_7\xi)$, so that σ interchanges ρ_s with $\rho_{7+s}\xi$ and $\bar{\rho}$ is the average of $\rho_1, \cdots, \rho_{14}$. Then we have $\langle \rho_s, \rho_t \rangle$ is equal to $\sqrt{2}$ or 0 according to whether the two nodes are joined or not joined in the diagram D. We have, for $s = 1, \cdots, 14$,

$$\langle \bar{\rho}, \rho_s \rangle = |\bar{\rho}|^2 = 1/(2 + 3\sqrt{2}) \tag{9}$$

From (3) and (4) we get

$$|w_{\mathcal{P}}|^2 = |w_{\mathcal{L}}|^2 = 2 \text{ and } \langle \bar{\rho}, w_{\mathcal{P}} \rangle = \langle \bar{\rho}, w_{\mathcal{L}}\xi \rangle = 1/\sqrt{2} \tag{10}$$

5.2 The height of a root

We use the Weyl vector $\bar{\rho}$ to define the *height of a root r* as

$$\mathrm{ht}(r) = |\langle \bar{\rho}, r \rangle|/|\bar{\rho}|^2$$

The 14 roots of the diagram D have height equal to 1. Take the roots of the reflections generating $R(L)$, found in 4.4, given in the coordinate system $\Lambda \oplus H$. Using the explicit isomorphism found in 2.5 we write them in the coordinate system $3E_8 \oplus H$. We use the above definition of height and run a "height reduction algorithm" (see theorem 5.7 in [3]) on the 81 generators for $R(L)$ found before to see that one can always get to an unit multiple of an element of D. (sometimes one needs to perturb if the algorithm gets stuck - at most one perturbation was enough in all cases). This proves

5.3. Theorem. *The order 4 reflections in the roots of D generate the reflection group of L.*

Moreover, as in lemma 3.2 of [3] we can show that the reflections in the roots of the M_{444} diagram generate the reflection group of L. We just check that the relation deflate(y) holds for each octagon y inside D. For example, if $y = (d_1, c_1, b_1, a, b_2, c_2, d_2, e_3)$, the relation deflate$(y)$ follows from the equation $\phi_{d_1}^i \phi_{c_1}^i \phi_{b_1}^i \phi_a^i \phi_{b_2}^i \phi_{c_2}^i(d_2) = -e_3$.

Now we prove the analog of Proposition 6.1 of [3]. The proof is also exactly similar.

5.4. Proposition. *The 14 roots of the diagram D are the only roots (up to units) having the minimum height 1. All other roots have strictly bigger height. (In other words, the mirrors of the roots in D are the 14 mirrors closest to the vector $\bar{\rho}$).*

Proof. We need the following distance formulae for the metric on the quaternionic hyperbolic space $\mathbb{H}H^n$ (See [9]). A positive norm vector x in the vector space determines a point in the hyperbolic space, also denoted by x. A negative norm vector r determines a totally geodesic hyperplane given by $r^\perp$. Let $c(u,v)^2 = \frac{|\langle u,v \rangle|^2}{|u|^2 |v|^2}$. Then we have

$$\cosh^2(d(x, x')/2) = c(x, x')^2, \qquad \sinh^2(d(x, r^\perp)/2) = -c(x, r)^2 \tag{11}$$

Two hyperplanes $r^\perp$ and $r'^\perp$ meet in the hyperbolic space if $c(r, r') < 1$, are asymptotic if $c(r, r') = 1$ and do not meet if $c(r, r') > 1$ in which case the distance between the hyperplanes is given by

$$\cosh^2(d(r^\perp, r'^\perp)/2) = c(r, r')^2 \tag{12}$$

Let r be a root of the lattice L with $\mathrm{ht}(r) = |\langle \bar{\rho}, r\rangle|/|\bar{\rho}|^2 \leq 1$. We want to prove that r is a unit multiple of one of the 14 roots of D.

Let x be a point in D. Either $|\langle x, r\rangle| \leq 2$, or using the triangle inequality $d(r^\perp, x^\perp) \leq d(r^\perp, \bar{\rho}) + d(x^\perp, \bar{\rho})$ along with the distance formulae (11) and (12) above we get

$$|\langle x, r\rangle| \leq 2\cosh(2\sinh^{-1}(|\bar{\rho}|/\sqrt{2})) \approx 2.32$$

So we must have $|\langle x, r\rangle|^2$ equal to 0, 2, or 4.

Similarly from $d(r^\perp, w_{\mathcal{P}}) \leq d(r^\perp, \bar{\rho}) + d(w_{\mathcal{P}}, \bar{\rho})$, (11) and (12) we get

$$|\langle w_{\mathcal{P}}, r\rangle| \leq 2\sinh(\sinh^{-1}(|\bar{\rho}|/\sqrt{2}) + \cosh^{-1}(1/2\bar{\rho})) \approx 2.26$$

It follows that $|\langle w_{\mathcal{P}}, r\rangle|^2$ is equal to 0, 2, or 4.

We can write

$$r = \sum_{x \in \mathcal{P}} -x\langle x, r\rangle/2 + w_{\mathcal{P}}\langle w_{\mathcal{P}}, r\rangle/2 \tag{13}$$

taking norm in (13) we get

$$-2 = \sum_{x \in \mathcal{P}} -|\langle x, r\rangle|^2/2 + |\langle w_{\mathcal{P}}, r\rangle|^2/2 \tag{14}$$

There are only few cases to consider. Multiplying r by a unit, we may assume that $\langle w_{\mathcal{P}}, r\rangle$ is either 0 or p or 2. In the following let u_1, u_2 etc denote units in $\mathcal{H}^*$ and x_1, x_2 etc. denote points of D.

If $\langle r, w_{\mathcal{P}}\rangle = 0$, from (14) we get $\sum |\langle x_s, r\rangle|^2 = 4$. Then the unordered tuple $(\langle x_1, r\rangle, \cdots, \langle x_7, r\rangle)$ is equal to $(2u_1, 0^6)$ or $((pu_1, pu_2, 0^5)$. So either r is an unit multiple of x_s (in which case it has height equal to one) or $r = (x_1pu_1 + x_2pu_2)/(-2)$. Using diagram automorphisms (which is 2-transitive on points of D) we can assume that $x_1 = a$ and $x_2 = c_1$ and then check that there is no such root r.

If $\langle w_{\mathcal{P}}, r\rangle = p$, then $\sum |\langle x_s, r\rangle|^2 = 6$; the unordered tuple $(\langle x_1, r\rangle, \cdots, \langle x_7, r\rangle)$ is equal to $(2u_1, u_2p, 0^5)$ or $(u_1p, u_2p, u_3p, 0^4)$. In the first case we get $r = 2x_1u_1/(-2) + x_2u_2p/(-2) + w_{\mathcal{P}}p/2$. Taking inner product with $\bar{\rho}$ and using $\langle \bar{\rho}, w_{\mathcal{P}}\rangle/|\bar{\rho}|^2 = 3 + \sqrt{2}$ we get $\langle \bar{\rho}, r\rangle/|\bar{\rho}|^2 = -u_1 - u_2/\bar{p} + (3 + \sqrt{2})/\bar{p}$ which clearly has norm greater than one.

In the second case we get $r = \sum_{s=1}^{3} x_s u_s p/(-2) + w_{\mathcal{P}}p/2$ which implies $\langle \bar{\rho}, r\rangle/|\bar{\rho}|^2 = (-u_1 - u_2 - u_3 + 3 + \sqrt{2})/\bar{p}$. Again this quantity has norm at least one. We now show that the only way it can be equal to one is if r is a unit multiple of $l_1, \cdots, l_7$.

The only way one can have $\mathrm{ht}(r) = 1$ in the above paragraph is if r has inner product p with three of the points x_1, x_2, x_3 and orthogonal to others. If x_1, x_2, x_3 do not all lie on a line then there is a line l that avoids all these three points. Taking inner products with r in the equation $w_{\mathcal{P}} = l\bar{p} + \sum_{x \in l} x$ gives $p = p\langle l, r\rangle$ contradicting $L'p = L$. So x_1, x_2, x_3 are points on a line l_1. It follows that r and an unit multiple of l_1 has the same inner product with each element of $\mathcal{P}$ and with $w_{\mathcal{P}}$. So r is an unit multiple of l_1.

If $\langle w_{\mathcal{P}}, r\rangle = 2$, and $\sum |\langle x_s, r\rangle|^2 = 8$, the unordered tuple $(\langle x_1, r\rangle, \cdots, \langle x_7, r\rangle)$ is equal to $(2u_1, 2u_2, 0^5)$ or $(2u_1, u_2p, u_3p, 0^4)$ or $(u_1p, \cdots, u_4p, 0^3)$. Using similar calculation as above, we get $\langle r, \bar{\rho}\rangle/|\bar{\rho}|^2$ is equal to $(-u_1 - u_2 + 3 + \sqrt{2})$ or $(-u_1 - u_2/\bar{p} - u_3/\bar{p} + 3 + \sqrt{2})$ or $(-(u_1 + \cdots + u_4)/\bar{p} + 3 + \sqrt{2})$ respectively. Again each of these quantities are clearly seen to have norm strictly bigger than one. $\qquad \square$

5.5 Remarks

1. The group generated by the Coxeter diagram $Inc(P^2(\mathbb{F}_2))$ when the vertices are made into reflections of order two is the group $O_8^-(2) : 2$ as was found by Simons in [10]. It would be interesting to understand the relation between these two groups in a more conceptual way.

2. The order of the "spider element" $sp = ab_1c_1ab_2c_2ab_3c_3$ in the reflection group $R(L)$ is 40.

3. Since we can find a M_{444} diagram in $L = \Lambda \oplus H$ with the 6 vectors in the three E_8 hands of the form $(\lambda; 1, *)$ with λ in the first shell of Leech lattice, it is likely that the reflection group of L in-fact equals the whole automorphism group. It would probably follow after a little more work in the line of Theorem 6.2 of [1] and using the fact that $\mathbb{T} \subseteq R(L)$. But I have not checked this.

4. Exactly as in Remark 3.4 of [3] the $\mathcal{H}$-lattice $L = 3E_8 \oplus H$ can be defined by the starting with a singular lattice corresponding to the vertices of the diagram D and quotienting out by the relations $l(ip) + \sum_{x \in l} x = l'(ip) + \sum_{x \in l'} x$.

5. The calculations needed for this paper are done using the gp calculator and the codes (for finding the explicit isomorphism from $3E_8 \oplus H$ to $\Lambda \oplus H$ and for the height reduction algorithm to show that the 14 nodes of D generate $R(L)$) are contained in the file

`quat.gp`

The programs are available on the my web-site http://www.math.berkeley.edu/~tathagat

References

[1] Allcock, J. A. *The Leech lattice and complex hyperbolic reflections.* Invent. Math 140 (2000), 283-301.

[2] Allcock, J. A. *New complex- and quaternionic-hyperbolic reflection groups.* Duke Math. J. 103 (2000), 303-333.

[3] Basak, T. *Complex Lorentzian Leech lattice and bimonster* Preprint 2005. Arxiv. math.GR/0508228. To appear in the Journal of Algebra.

[4] Conway, J. H. *The automorphism group of the 26-dimensional even unimodular Lorentzian lattice.* J. Algebra 80 (1983), no. 1, 159-163.

[5] Conway, J. H., Norton, S.P. and Soicher, L.H. *The bimonster, the group Y_{555}, and the projective plane of order 3.* Computers in algebra (Chicago, IL, 1985), 27–50, Lecture Notes in Pure and Appl. Math., 111, Dekker, New York, 1988.

[6] Conway, J.H. and Simons, C.S. *26 implies the bimonster* J. Algebra 235 (2001), 805-814.

[7] Conway, J.H. and Sloane, N.J.A. *Sphere packings, lattices and groups 3rd ed..* Springer-Verlag (1998)

[8] Ivanov, A.A. *Geometry of sporadic groups I. Peterson and tilde geometries* Encyclopedia of mathematics and its applications vol 76. Cambridge University press, 1999.

[9] Kim, I. and Parker, J.R. *Geometry of quaternionic hyperbolic manifolds* Math. Proc. Camb. Phil. Soc. (2003), 291-320.

[10] Simons, Christopher S. *Deflating infinite Coxeter groups to finite groups.* Proceedings on Moonshine and related topics (Montral, QC, 1999), 223–229, CRM Proc. Lecture Notes, 30, Amer. Math. Soc., Providence, RI, 2001

[11] Wilson, R.A. *The quaternionic lattice for $2G_2(4)$ and is maximal subgroups.* J. Algebra 77 (1982), 449-466.